Nuclear reactions of astrophysical interest

- Introduction
- Scales of energy and density
- Reaction rate
- Neutron induced processes
- Cross sections of charged particles
- Astrophysical factors
- The Gamow peak and the reaction rate
- Reaction rates and temperature

Introduction

- Most important processes are neutron induced reactions:

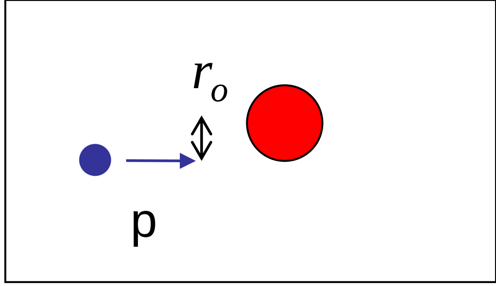
$$n + {}^{A}Z \rightarrow {}^{A+1}Z + \gamma$$

- and reactions between nuclei

$${}^{A1}Z_1 + {}^{A2}Z_2 \rightarrow {}^{A}Z + X$$

- The first reactions are relevant when one has free neutrons, as in the big bang and /or in advanced stellar phases where neutrons are produced , (e.g.. ${}^{22}Ne + {}^{4}He \rightarrow {}^{25}Mg + n$) or from neutronization reactions
- Reaction between nuclei are important for forming heavier elements,as in the big bang and/or in stellar combustion

Scales of energy and angular momentum



- The collision energies of interest to us are generally smaller than the typical nuclear energy scale, order 1MeV:
- Indeed, primordial nucleosysnthesis becomes effective at kT<100 keV, otherwise the first products (deuterons) disintegrate through photodissociation
- Nucleosynthesis in the Sun occurs at kT ≈ 1keV; Helium burning stars have KT ≈10KeV
- Typical momenta are thus $p \approx (kT\ m)^{1/2}$; Put mc^2=1GeV e kT=100 KeV . One gets p ≈ 10 MeV/c
- Since the typical nuclear dimensions are $r_0 \approx$ 1fm the angular momentum involved for the nuclear collision is, classically,

$$L = p\ r_0 = 10(\mathrm{MeV/c\ fm}) \approx (1/20)\ \hbar,$$

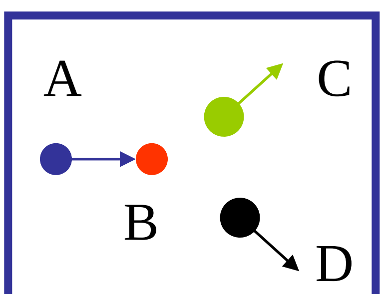
thus the dominant processes are dominated by the smallest angular momenta, compatible with quantum mechanics.

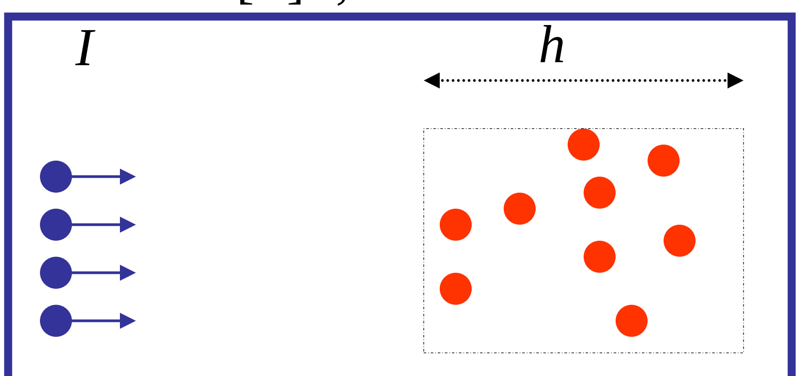
Note I: cross section

- The cross section is the main statistical quantity which characterizes a collision process.
- Suppose you study A+B→C+D by shooting a beam of particle A onto a target of B
- The beam is characterized by the "current " *I* (number of particles produced per unit time), by the particle type and energy.
- The target is characterized by the target density (number /volume) *n* and by the thickness *h*.
- The measurable quantity is the number of reactions per unit time $\Delta N/\Delta t$.
- For a thin target (i.e. Such that there is a small probability of having one interaction per incident particle) the number of reactions per unit time will be proportional to the current, density and thickness :

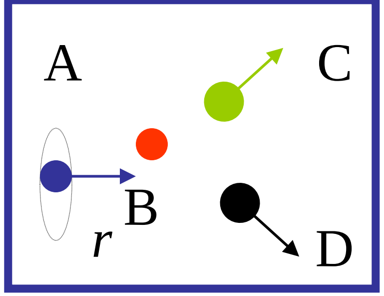
$$\Delta N/\Delta t = I\, n\, h\, \sigma.$$

- The proportionality constant, having dimensions $[L]^2$, is the cross section for reaction A+B->C+D





Note II: more on the cross section



- Suppose that A+B-> C+D occurs when particles A e B are at distance d<r, with probability w.
- For each particle (i) entering the target, the probability P_i of having a reaction will be given by the number of encounters N_i occurring with distance d <r, multiplied by the probability w in each encounter, $P_i = N_i w$ *.
- When averaging over many particles

  $N_i \rightarrow \langle N \rangle = \pi r^2 n h$

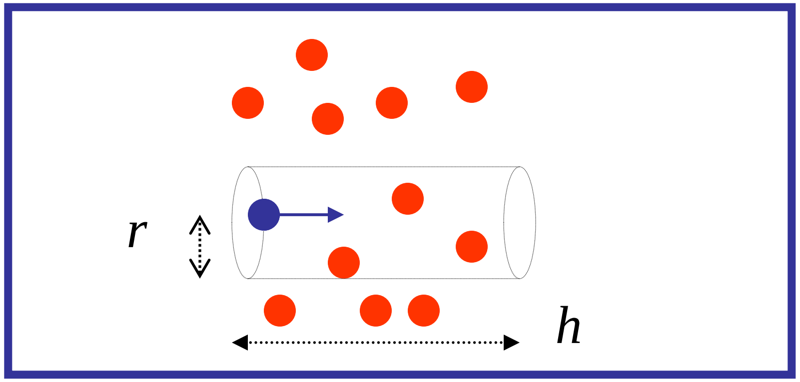
  and $P_i \rightarrow \langle P \rangle = \pi r^2 w n h$
- If I is the number of particles entering the target per unit time, the number of reactions per unit time will be I<P> and thus :

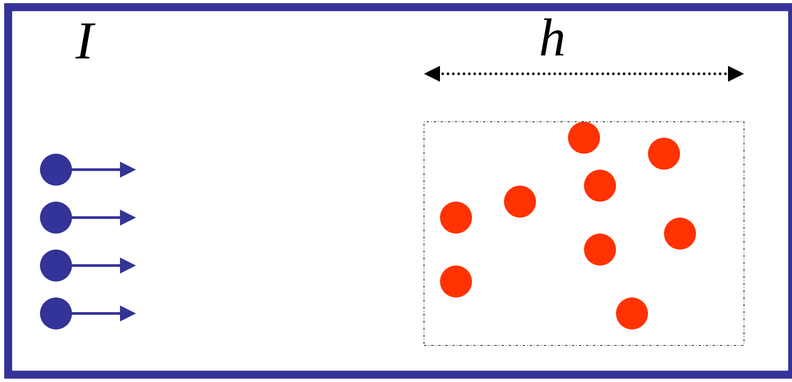
  $$\Delta N/\Delta t = I n h \pi r^2 w$$
- By comparing with the previous definition of σ

  $$\Delta N/\Delta t = I n h \sigma$$
- We get

  $$\boldsymbol{\sigma = \pi r^2 w}$$

*This holds for a "thin target"



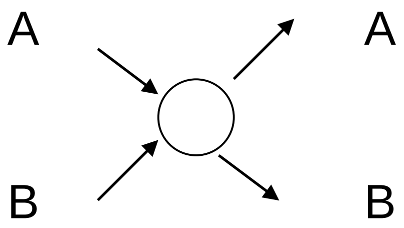


- **The cross section is thus the product of the geometrical area where the reaction can occur multiplied with the probasbility of occurrence**

Elastic cross sections in quantum mechanics

- We are interested in collision processes to be described by quantum mechanics in the limit of low energy .
- Generally , the wave function Ψ(r) which describes the relative motion of the two particles obeys to a multi-component Schroedinger equation, corresponding to the different reaction channels (elastic, inelastic…).
- When **only** the elastic scattering is possibile , there is a single component, which satisfies :

$$-\frac{\hbar^2}{2m}\Delta\Psi + V(r)\Psi = E\Psi$$



- Where $m=m_A m_B/(m_A+m_A)$, E is the collision energy, i.e. the energy in the c.m. system and the asymptotic conditions are given by

$$\Psi_{as} = e^{ikz} + f(\vartheta)\frac{e^{ikr}}{r}$$

with $k^2 \hbar^2 = 2mE$

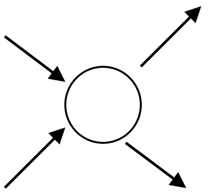
- The differential cross section per unit solid angle is

$$d\sigma / d\Omega = \left| f(\theta) \right|^2$$

- And the integrated elastic cross section is

$$\sigma = 2\pi \int \sin\theta d\theta \left| f(\theta) \right|^2$$

Behavior of the low energy elastic cross section



- The wave function can always be expanded in eigenfunctions of the angular momentum
- For a potential V(r) the orbital angular momentum is conserved .
- The expansion in spherical armonics corresponds to an expansion of the scattering amplitude

$$f(\vartheta) = (1/2ik)\sum_l (2l+1) f_l P_l(\cos\theta)$$

- In terms of the partial amplitues *f_l*, due to the orthogonality of the eigen functions of the angular momentum the integrated corss section is

$$\sigma = \sum \sigma_l = \sum 4\pi(2l+1)\left|f_l\right|^2$$

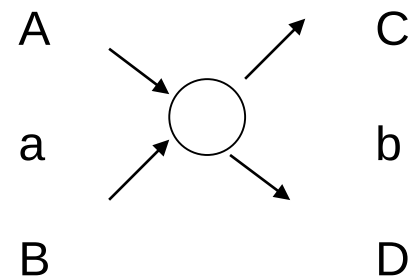
- One can show that in the low energy limit for potentials decreasing sufficiently fast with r*, f_l is proportional to k^{2l} and thus the dominant contribution arises from l=0.
- It follows that for k->0:

A) the differential cross section is isotropic

B) the elastic cross section tends to a constant

*)*The coulomb case, V=1/r does not respect this condition*

Cross section for inelastic processes



- Consider for simplicity two body reactions

  $A+B \rightarrow C+D$

- We remind that exo-energetic are those reactions such that

  $M_A+M_B > M_C+M_D$

- Whereas one has endo-energetic reactions for

  $M_A+M_B < M_C+M_D$

- The problem can be described by a multicomponent wave function $\Psi = (\Psi_a, \Psi_b, \ldots)$ with boundary conditions

  $\Psi_a = \exp(ik_a z) + f_{aa}(\theta)\exp(ik_a r)/r$ in the elastic channel (a)

  $\Psi_b = f_{ab}(\theta)\exp(ik_b r)/r$ in the other channels ($b \neq a$)

  where k is the momentum associated with the relative motion of the two particles for a fixed energy E:

  $k_a^2 \hbar^2 = 2m_a E = 2E m_A m_B/(m_A+m_B)$

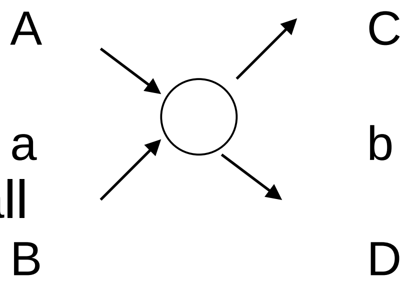
- The differential cross sections are

  $$d\sigma_{ab} = |f_{ab}|^2 \, p_C/p_A \, d\Omega_C$$

  where p_A (p_C) is the momentum of the projectile (or the produced) particle respectively.

$d\sigma_{ab}=|f_{ab}|^2\, p_C/p_A\, d\Omega_C$

Behaviour of the inelastic cross section at low energy



- In the low energy limit only the s-wave contributes and all amplitudes f_{ab} tend to constant values*.
- From the foregoing equation , $d\sigma_{ab}=|f_{ab}|^2\, p_C/p_A\, d\Omega_C$ one has :

  A) for exothermic reactions, wihich can occurr at arbitrarly small energy , since f_{ab} tends to a constant and p_C to a finite limit**,

  $$\sigma_{in} = \text{cost } v^{-1}$$

  B) for endothermic reactions, there is a threshold energy Δ. The process can only occur if $E=\frac{1}{2}\, k_a m_a^{\,2} > \Delta$ and the momentum of the relative motion in the final state is given by $\frac{1}{2}\, k_b m_b^{\,2} = E-\Delta$. The momentum p_C' of each particle in the final state is proportional to k_b and thus to $(E-\Delta)^{1/2}$. This implies that in a process in S-wave the behavior of cross sections in proximity of a threshold is

  $$\sigma_{in} = \text{cost } (E-\Delta)^{1/2}$$

- ** again for a potential decreasing sufficiently fast with distance.*
- ***Note that a similar argument gives* σ_{el} *cost*

Cross sections and reaction rates

- The cross section expresses the probability of interaction for a fixed collision energy. In a plasma, in general energy is not fixed, but there is a distribution of collision energies, and the reaction rate is a more useful concept.
- If one has n_A and n_B particles of type A and B , distinguishable*, per unit volume , and if v is their relative speed, the number of interactions per unit volume and time is

$$r = n_A n_B \sigma_{AB}(v)v$$

- If f(v) is the distribution of relative velocities, [$\int_o dvf(v)=1$], the number of interactions per unit volume and time is

$$R = \langle r \rangle = n_A n_B [\int_o dv\, \sigma_{AB}(v)vf(v) = n_A n_B \langle \sigma_{AB}v \rangle$$

where $\langle \sigma_{AB}v \rangle$ is defined as the reaction rate per couple of particles.

*****_If patrticles are identical, one cannot distinguish AB da BA and a factor ½ has to be inserted when evaluating r_

The nuclear velocity distribution

- The Maxwell Boltzmann distribution is generally adequate for the nuclear motion since:
- A) Thermalization times are short with respect to nuclear reaction times
- B) nuclear speeds are non relativistic
- C) quantum effects are negligible
- D) interaction energies are generally negligible with respect to kinetic energies

- C) comes from the fact that the phase space per particle

  $\Phi = \int_o d^3p\, d^3q \approx (mkT)^{3/2}\, n^{-3}$

  is large with respect to the dimension of the unit cell of phase space , h^3.

  (Note that this condition is satisfied for nuclei, not for electrons)
- D) corresponds to the assumption of a perfect gas, $\langle T\rangle \gg \langle V\rangle$.
- This condition can hold also for a dense plasma, e.g. in the Sun kT≈ keV and V ≈ 10 eV

The Maxwell Boltzmann distribtion

- Assume that a system at thermodinamical equilibrium with temperature T is described by

  $E=p^2/2m + U(\mathbf{r})$

  The probability distribution in phase space is

  $dP/d^3p d^3r = A \exp(-E/kT)$

  where A is a normalization constant
- For a perfect gas U=0 and thus:

  $dP/d^3p d^3r = A \exp(-p^2/2mkT)$
- The momentum distribution is obtained by integrating over volume

  $dP/d^3p = AV \exp(-p^2/2mkT)$

  where the new constant B=AV is determined by requiring:
- $\int dP/d^3p = 1$
- The calculation of the Gaussian integral (see appendix) determines AV and thus

  $dP/d^3p=(2\pi mkT)^{-3/2} \exp(-p^2/2mkT)$
- From $d^3p=4\pi p^2dp$ one can get the distribution as a function of p:

  $dP/dp = 4\pi p^2(2\pi mkT)^{-3/2}\exp(-p^2/2mkT)$
- From $E=p^2/2m$ one has $dE=pdp/m$ and thus one can get the energy distribution

  $dP/dE= m4\pi p(2\pi mkT)^{-3/2}\exp(-p^2/2mkT)$
- This means

  $\mathbf{dP/dE= 2/\pi^{1/2} (kT)^{-3/2} E^{1/2} \exp(-E/kT)}$
-

Properties of the MB distribution

- The important result is the energy distribution

$dP/dE = 2/\pi^{1/2}(kT)^{-3/2}E^{1/2}\exp(-E/kT)$

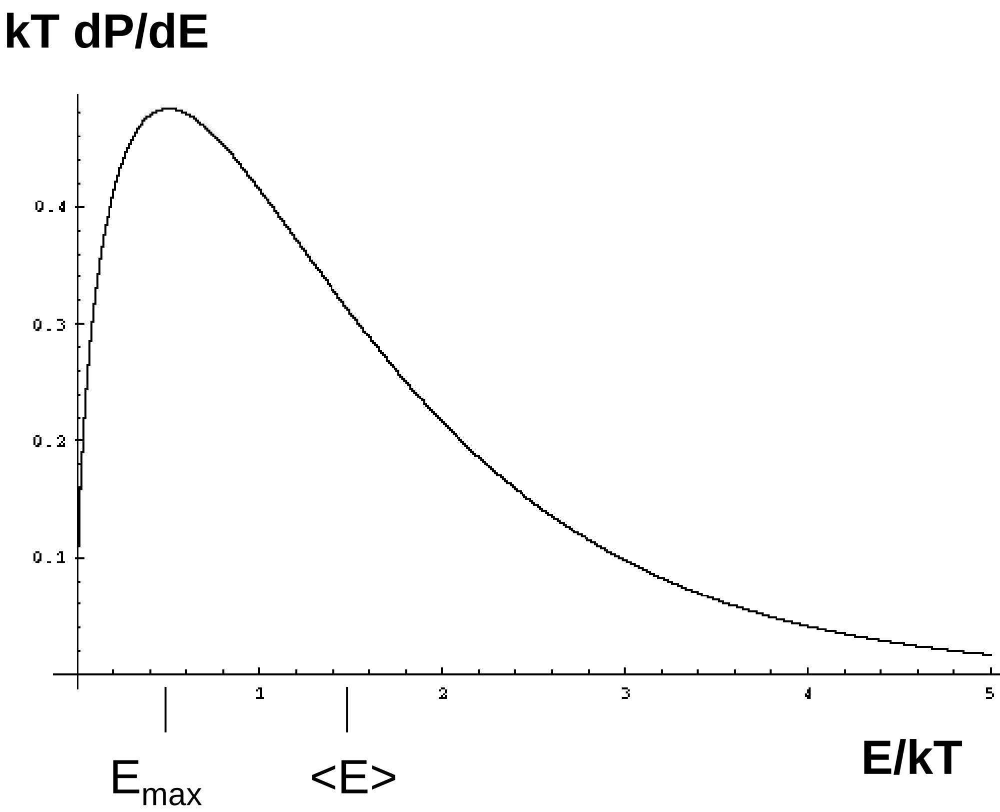
- One can check that :

  i) the normalization condition $\int dP/dE=1$ is verified

  ii) energy equipartition holds

  $\langle E\rangle=3/2\ kT$

  iii) the most probable energy is $E_{max}= ½\ kT$



The distribtion of relative velocities and of collision energies

- If we consider two non interacting particles, A and B, their hamiltonian is

  $$H = p_A^2/2m_A + p_B^2/2m_B + U(\mathbf{r})$$

  We can describe the system by using the CM coordinate **R**, the relative distance **r** and the conjugated momenta

  $\mathbf{P} = p_A + p_A$; $\mathbf{p} = m\mathbf{v}$

  where m is the reduced mass and v is the relative velocity
- In terms of these variables one has for the Hamiltonian

  $$H = \mathbf{P}^2/2m_{tot} + p^2/2m + U(\mathbf{r}) =$$
  $$= H(\mathbf{P}) + h(\mathbf{p},\mathbf{r})$$

- The hamiltonian has been factorized in two terms

  A) the first describes the motion of the CM

  B) The second expresses the energy of the collision.
- Form the thermodynamical point of view, the two pieces can be treated independently
- For the relative motion of the nuclei, one needs to study only h(**p,r**).
- In the approximation of perfect gas, we thus have for the collision energy **E**

  $$dP/dE = 2/\pi^{1/2} (kT)^{-3/2} E^{1/2} \exp(-E/kT)$$

Reaction rates and lifetimes averaged over the MB distribution

- Since cross sections are generally expressed in terms of the collision energy, $E= ½ m v^2$, it is convenient to express average quantities in terms of energy and not speed. This can be done by a change of variables v→E

  (1) $<\sigma v> = (8/\pi m)^{1/2} (kT)^{-3/2} \int_o dE\ \sigma(E)\ E \exp(-E/kT)$

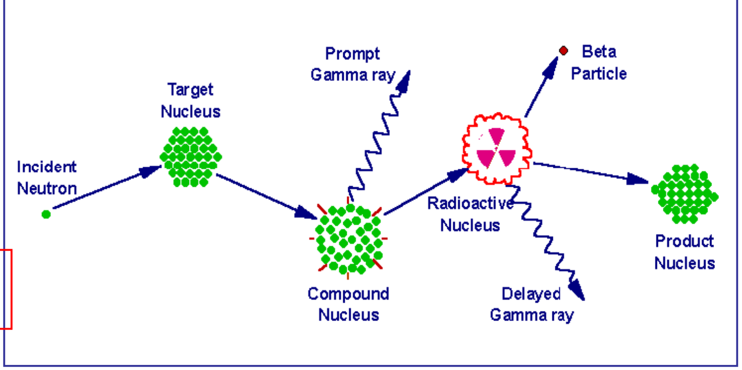
- Eq (1) is essential for establishing the chemical evolution of a plasma. In the following pages we shall apply (1) to the case of neutrons and of charged particles.
- Let us note that the integrand has a maximum at an energy which is the most likely for the process; this does not need to coincide with the most likely energy in the particle distribution,
- We remark that a nucleus, stable if isolated, becomes unstable when nuclear reactions are possible. If the only available process is A+B→ C+D the number density of A nuclei evolves according to

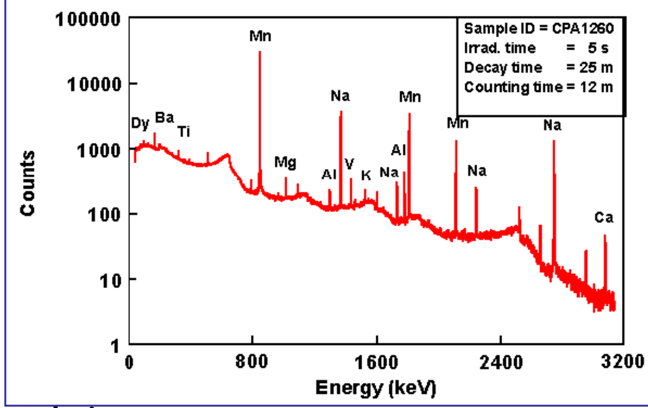
  $$dn_A/dt= -n_A n_B <\sigma v>$$

  thus the population of nuclei A decreases exponentially, with a decay rate $\lambda_A= n_B <\sigma v>$. Its inverse represents the lifetime of the nucleus A in the specified conditions.

Notes on neutron capture

$n + (Z,A) \rightarrow (Z,A+1) + \gamma$





- Neutron capture is the way of stabilizing neutrons inside nuclei
- The free neutron decays $n \rightarrow p + e + \text{anti-}\nu$ with $=885.7\pm0.8$ s. This process is allowed since $Q_n = m_n - m_p - m_e = 0.7$MeV is positive
- The deuteron d=(p,n) is stable , thus the neutron inside the nucleus is stable.
- The process $d=(p,n) \rightarrow p + p + e + \text{anti-}\nu$ cannot occur since it would violate energy conservation , in fact $Q_d = m_d - 2m_p - m_e = -E_b + Q_n = -2.2+0.7 = -1.5$MeV is negative
- The binding energy of the nucleus prohibits the neutron decay, in the deuteron as well as in other stable nuclei.
- Neutron capture on nuclei provides gamma radiation specific of the compound nucleus which is being formed.
- **$n + (Z,A) \rightarrow (Z,A+1)^* + \gamma_1$;**

  $(Z,A+1)^* \rightarrow (z,A+1) + \gamma_2$
- This is a method to study experimentally the energy levels of nuclei (neutron spectroscopy) and provides a method for the analysis of elements present in a substance, also at very low levels (Neutron Activation Analysis, NAA)*

*See http://en.wikipedia.org/wiki/Neutron_activation_analysis

Neutron capture

- Consider a reaction $n + {}^{A}Z \rightarrow {}^{A+1}Z + \gamma$ where ${}^{A+1}Z$ is a stable nucleus, or anyway $m(n)+ m({}^{A}Z) > m({}^{A+1}Z)$.
- This reaction is exo-themal, so that at low energy σv= constant
- In the plasma, the energy of particles which give the largest contribution corresponds to the most likely energy E_T=1/2kT
- The knowledge of the cross section for neutron absorption on a nucleus A at energy E_T (σ_A) is enough to determine :

$$< \sigma v >= \sigma v= \sigma_A (2E_T/m)^{1/2}$$

- Let us consider nuclei in an environment with a neutron density *n.* If the only relevant process is neutron capture, then the densities of nuclear species evolve as

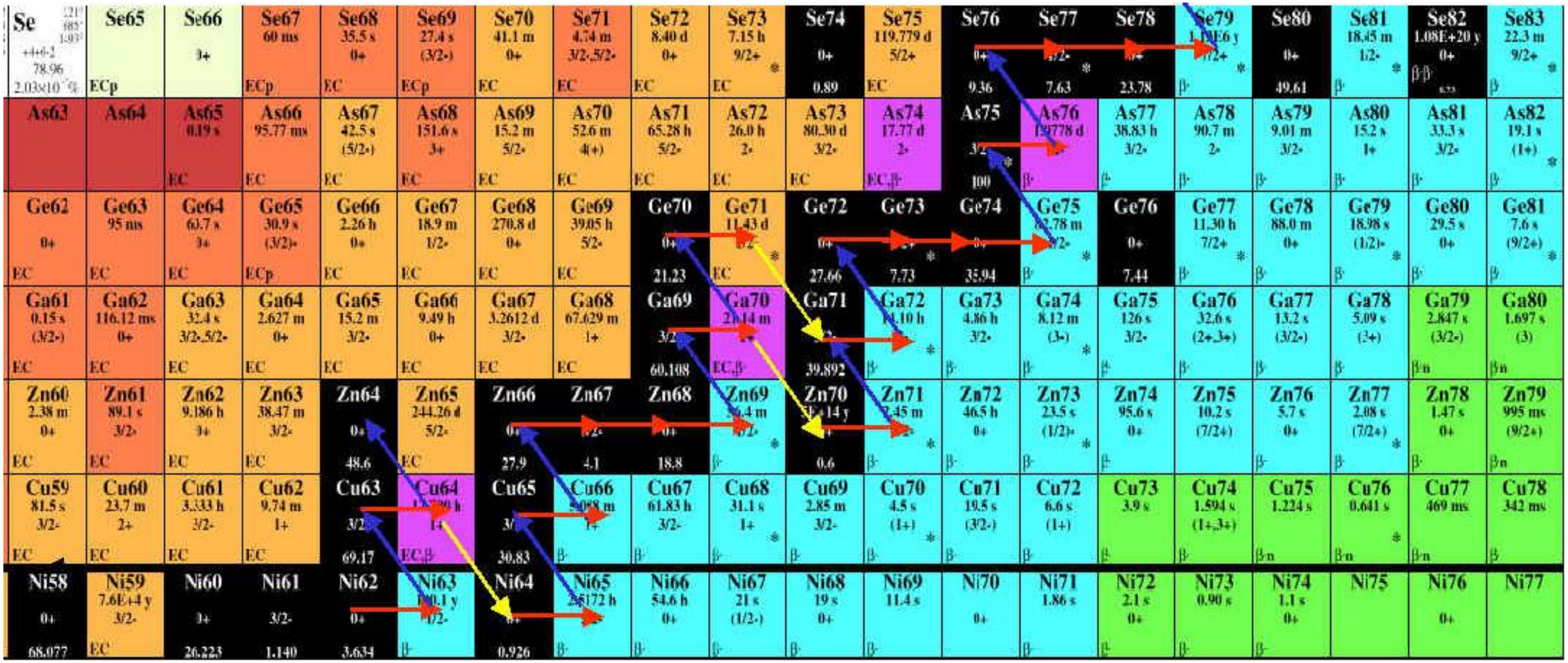
$$dn_A/dt = n <\sigma v>_{A-1} n_{A-1} - n<\sigma v>_A n_A$$

- The equilibrium condition $dn_A/dt=0$ corresponds thus to

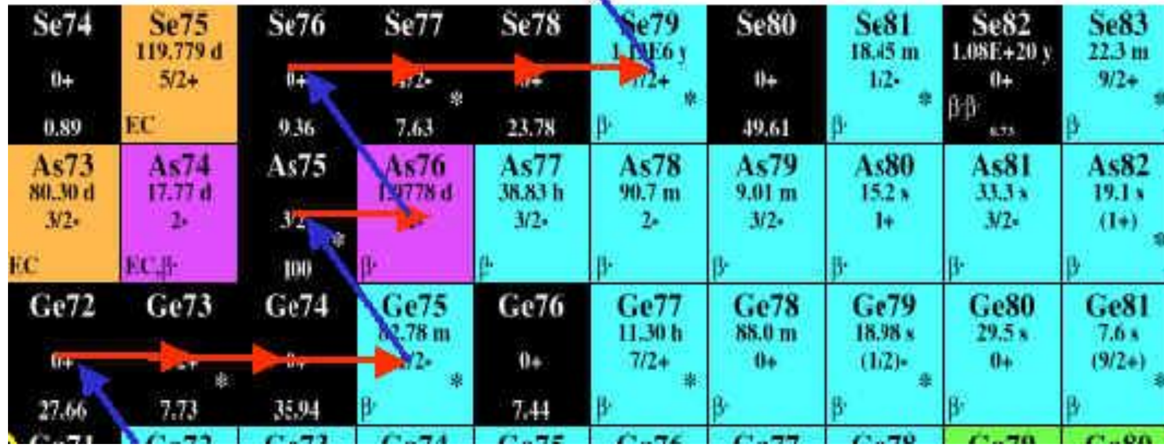
$$n_{A-1} \ \sigma_{A-1} = n_A \ \sigma_A$$

- This means that the density of the different species are inversely proportional to the respective cross sections

Netron capture and the formation of heavy elements



Heavy element distribution in the s-process (I)



- An important application of these concepts is to the case of slow absorption, i.e. those processes where neutron capture is slow with respect to decay .
- The equation for the number densities of the nuclei (Z,A) is generally more complicated than before

$$dn_{Z,A}/dt = n <\sigma v>_{A-1} n_{Z,A-1} - n<\sigma v>_{A} n_{,Z,A} - \lambda_Z n_{,Z,A} + \lambda_{Z-1} n_{,Z-1,A}$$

where the last terms represent the contribution of the beta decay

- Without these contributions, i.e. for stable nuclei, we go back to the previous equation
- If the decays are faster than capture, than for any A substantially only stable nuclei are formed
- In the plane (N,Z) the nuclear species which are produced are described by a zig zag line …..

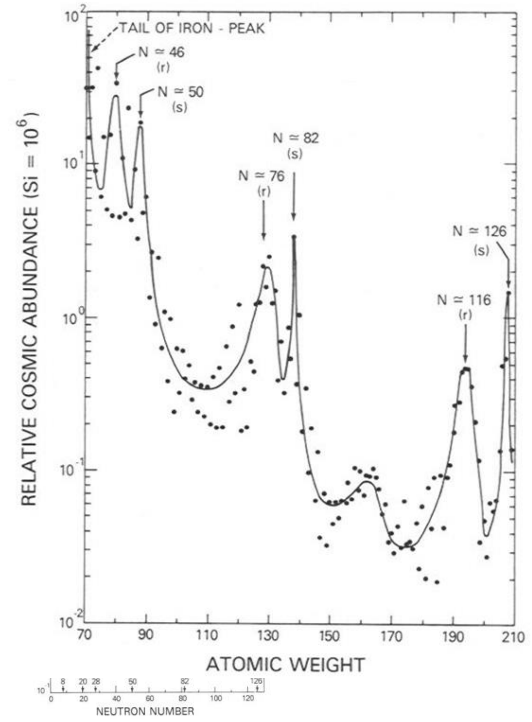
Elements produced by means of the s- process

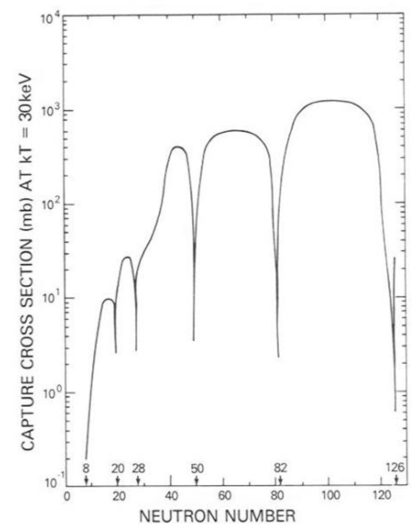
- In conclusion, in the hypothesis that nuclei are stable, or that nuclear decay is slow with respect to beta decay, one can eliminate the last terms, and the previous equation is again obtained

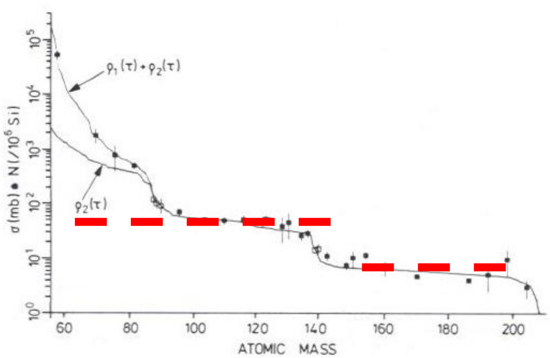
  $dn_A/dt = n \langle\sigma v\rangle_{A-1} n_{A-1} - n\langle\sigma v\rangle_A n_A$ which has to be interpreted as the equation for the density of stable nuclei, f

- In fact the observed abundances for heavy nuclei produced thorugh s-process satisfy to

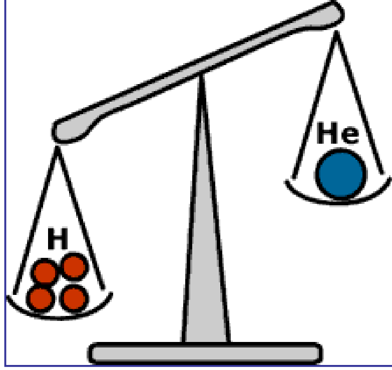
$$n_A \ \sigma_A = \text{constant}$$







Collisions between charged particles: the birth of nuclear astrophysics*



- In 1919, Henry Norris Russell, the leading theoretical astronomer in the United States, summarized in a concise form the astronomical hints on the nature of the stellar energy source. Russell stressed that the most important clue was the high temperature in the interiors of stars.
- F. W. Aston discovered in 1920 the key experimental element in the puzzle. He made precise measurements of the masses of many different atoms, among them hydrogen and helium. Aston found that four hydrogen nuclei were heavier than a helium nucleus. This was not the principal goal of the experiments he performed, which were motivated in large part by looking for isotopes of neon.
- The importance of Aston's measurements was immediately recognized by Sir Arthur Eddington, the brilliant English astrophysicist. Eddington argued in his 1920 presidential address to the British Association for the Advancement of Science that Aston's measurement of the mass difference between hydrogen and helium meant that the sun could shine by converting hydrogen atoms to helium.
- This burning of hydrogen into helium would (according to Einstein's relation between mass and energy) release about 0.7% of the mass equivalent of the energy. In principle, this could allow the sun to shine for about a 100 billion years.
- In a frighteningly prescient insight, Eddington went on to remark about the connection between stellar energy generation and the future of humanity:
- *“If, indeed, the sub-atomic energy in the stars is being freely used to maintain their great furnaces, it seems to bring a little nearer to fulfillment our dream of controlling this latent power for the well-being of the human race---or for its suicide”*

- **http://www.nobel.se/physics/articles/fusion/sun_3.html*

Eddington , Jeans e Gamow

- There was a long running scientific argument between **Jeans** and **Eddington** over the mechanism by which energy was created in stars. **Jeans** favoured, incorrectly as it turned out, the theory that the energy was the result of contraction while **Eddington**, correctly of course, believed it resulted from a slow process of annihilation of matter.

- The next major step in understanding how stars produce energy from nuclear burning, resulted from applying quantum mechanics to the explanation of nuclear radioactivity. This application was made without any reference to what happens in stars. According to classical physics, two particles with the same sign of electrical charge will repel each other, as if they were repulsed by a mutual recognition of 'bad breath'. Classically, the probability that two positively charged particles get very close together is zero. But, some things that cannot happen in classical physics can occur in the real world which is described on a microscopic scale by quantum mechanics.

- In 1928, George Gamow, the great Russian-American theoretical physicist, derived a quantum-mechanical formula that gave a non-zero probability of two charged particles overcoming their mutual electrostatic repulsion and coming very close together. This quantum mechanical probability is now universally known as the "Gamow factor." It is widely used to explain the measured rates of certain radioactive decays.

Collision between charged particles

- In a collision between two nuclei Z_1 e Z_2 at nuclear distances there are strong interactions, whereas at larger distances only the Coulomb interaction remains

$$V_C(r) = Z_1 Z_2 e^2/r$$

- Coulomb repulsion at nuclear distances ($R_n = r_0 A^{1/3}$ with) $r_0 \approx 1fm$) is given by:

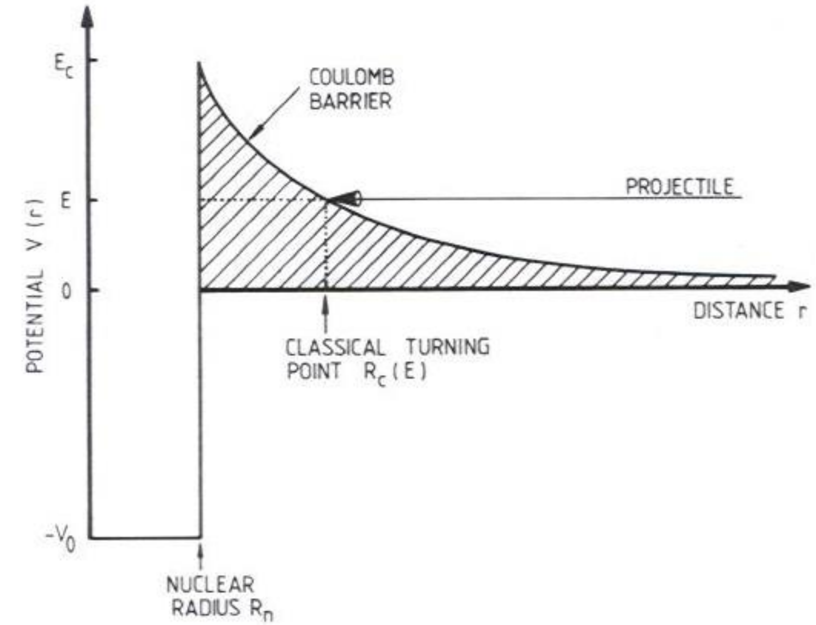
$$\varepsilon_C = V_C(R_n) = Z_1 Z_2 e^2/(r_0 A^{1/3})$$

- For light nuclei, the order of magnitude is

$$\varepsilon_C \approx e^2/r_0 = (e^2/\hbar c)\, \hbar c/r_0 =$$

$$(1/137)\ 200MeVfm/1fm = 1.4MeV$$

- Let us observe that in astrophysical plasmas thermal energies kT, are generally much smaller.



- According to classical physics, the shortest approach distance R_c for energy E is given by

$$R_c = Z_1 Z_2 e^2/E$$

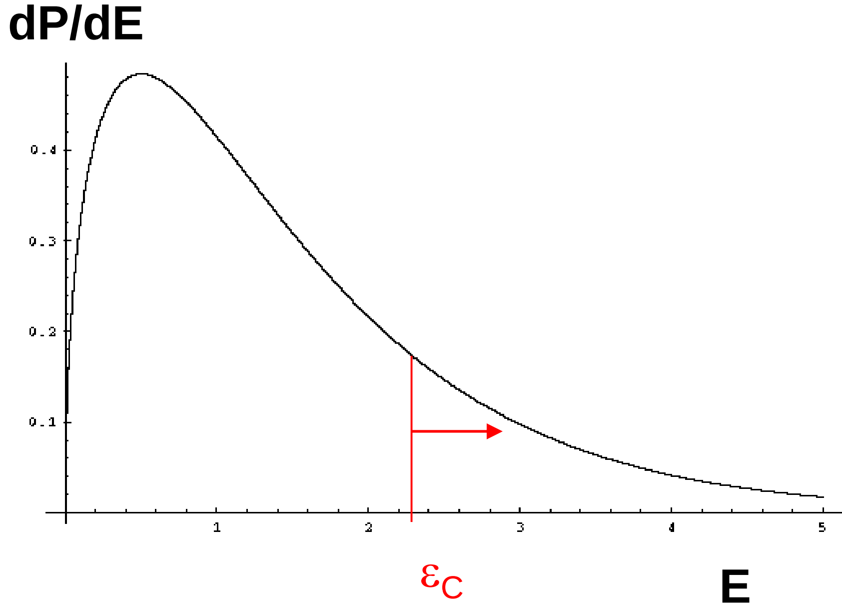
- Since this quantity is generally $>>R_N$, classically nuclear reactions cannot occur at thermal energies typical of plasmas
- In stars, as in the big bang, the possibility of nuclear reactions is strictly connected to the tunnel effect .

Remarks

- May be that the high energy tail in the Maxwell distribution accounts for the possibility of nuclear reactions ?
- The fraction of (couples of) particles with energy larger than ε_C is given by the integral of dP/dE beyond ε_C:
- This means :

$$F(\varepsilon_C) = 2\pi^{-1/2}(kT)^{-3/2}\int_{\varepsilon_C}^{\infty} dE\sqrt{E}\exp(-E/kT)$$

$$F(\varepsilon_C) = 2\pi^{-1/2}\int_{\varepsilon_C/kT}^{\infty} dx\sqrt{x}\exp(-x)$$



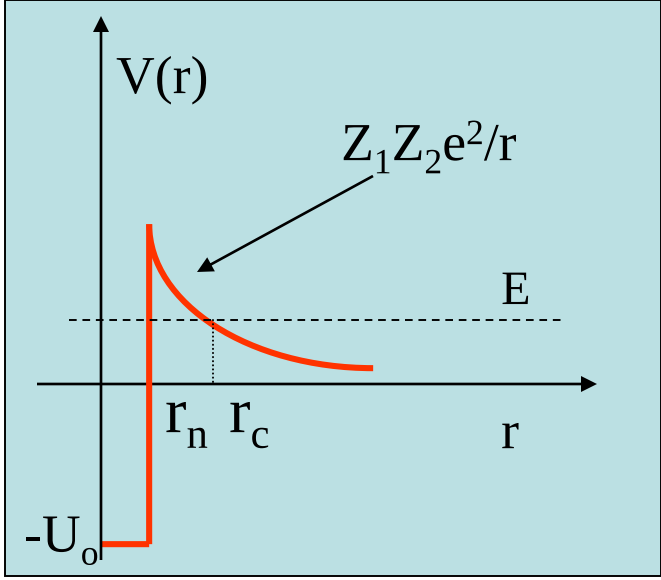
•Since in the integration region $x > \varepsilon_C/kT > 1$ one has $\sqrt{x} < x$ and thus

$$F(\varepsilon_C) < 2\pi^{-1/2}\int_{\varepsilon_C/kT}^{\infty} x\exp(-x)dx$$

$$= 2\pi^{-1/2}(1+\varepsilon_C/kT)\exp(-\varepsilon_C/kT)$$

•For $\varepsilon_C \approx 1$MeV e $kT \approx 1$ keV (as in the solar interior) $\varepsilon_C/kT \approx 1000$ and thus $F < 10^3 \exp(-1000) \approx 10^{-430}$, to be compared with the number of particles in the Sun 10^{57} (and in the whole universe 10^{70}) !!!

Penetration through the Coulomb barrier



- The problem for the two nuclei to reach nuclear distances through tunnel effect is very similar to the process of alpha decay, explained by Gamow in term of a transmission coefficient

  (1) $P = \exp(-2\pi Z_1 Z_2 e^2/\hbar v)$

  where $v = (2E/m)^{1/2}$ and P is known as the Gamow factor
- Note the exponential dependence on 1/v : the transmission probability is exponentially small when speed s are small
- We can write the equation as:

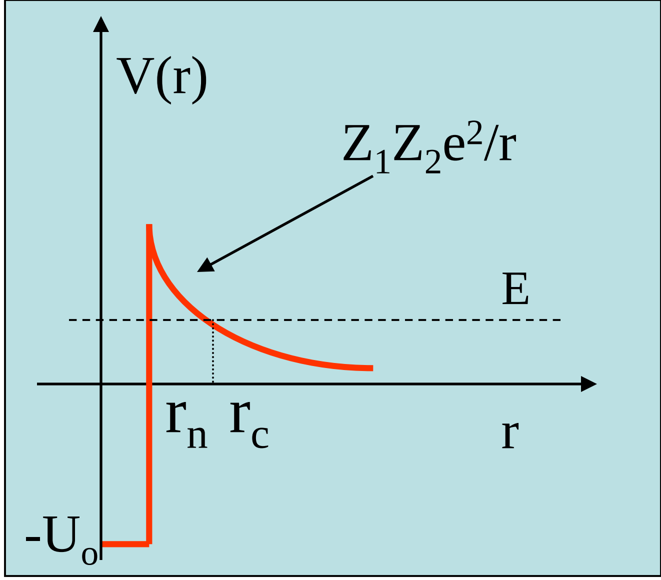
  $P = \exp(-2\pi v_c/\hbar v)$

- where the velocity scale is fixed by the Coulomb speed for two bodies with charge Z_1 e Z_2

  $v_c = Z_1 Z_2 e^2/\hbar$

- Eq (1) is obtained in a semiclassical approximation, neglecting nuclear size
- For a more precise treatment, look at Landa u QM and Rolfs,

The Coulomb energy scale



- We can express the probability in terms of the collision energy

  $P = \exp[-(E_C/E)^{1/2}]$

  where $E_C = \pi^2 m v_o^2$ is the Coulomb energy scale.
- If one writes explicitely

  $v_c = Z_1 Z_2 e^2/\hbar \quad m = m_n A_1 A_2/(A_1+A_2)$

  (where m_n is the nucleon mass)

  one has :

  $E_C = (Z_1 Z_2)^2 A_1 A_2/(A_1+A_2)\, U_C$

  with :

  $U_C = \pi^2 m_n c^2 \alpha^2 \approx 500$ keV

- Note that this scale is generally larger than kT.
- This has important consequences, since in collisions with E ≈ kT the probability of crossing the barrier is exponentially small

Fusion cross sections of charged particles

- Fusion cross section of nuclei contain three terms :
- 1) a term proportional to the de Broglie length square ($1/k^2$), and thus inversely proportional to the collision energy
- 2) the probability of tunnelling through the barrier P(E)
- 3) a term expressing the probability of nuclear interaction, once the barrier is passed .

- Cross sections have thus the form

$$\sigma(E) = \frac{S(E)}{E} \exp(-\sqrt{E_c / E})$$

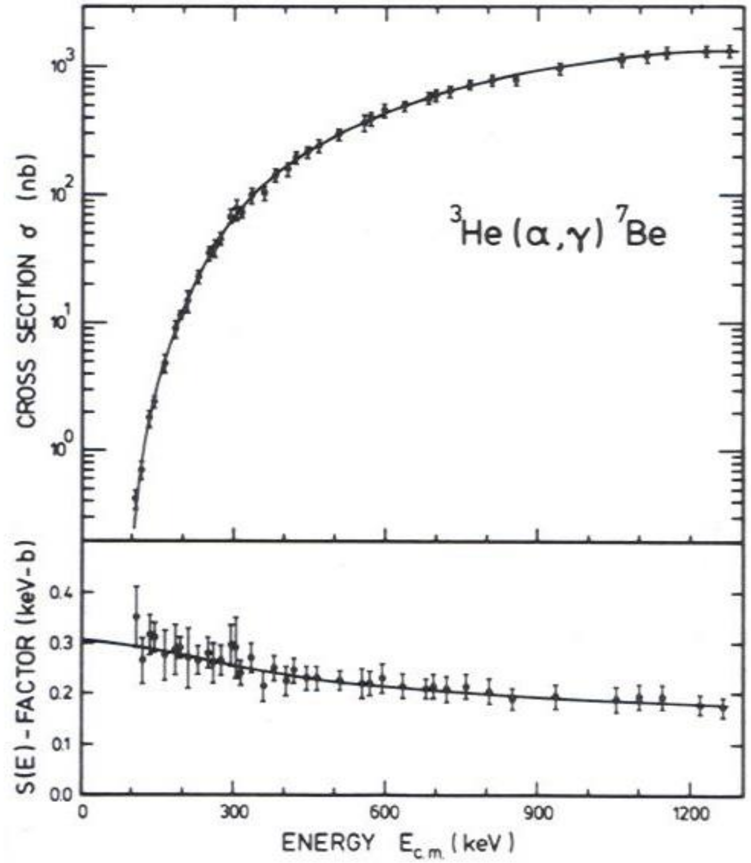
- The factor S(E) is called the astrophysical factor, and expresses the strength of the interaction
- Its dimensions are E L^2 , a commonly used unit being MeV barn

An example

$^{3}He + ^{4}He \rightarrow ^{7}Be + \gamma$

$$\sigma(E) = \frac{S(E)}{E} \exp(-\sqrt{E_c / E})$$

- At energies of order 1MeV
  $\sigma \approx 10^{-6}$barn $= 10^{-30}$cm^2
- The cross section decreases by 9 orders of magnitudes going to E =0.1 MeV
- This is a consequence of the exponential behaviour
- On the other hand, note that the astrophysical factor extracted from data is regular.
- This holds generally, unless there are resonances in the energy region of interest



Orders of magnitude of the astrophysical factor

$$\sigma(E) = \frac{S(E)}{E} \exp(-\sqrt{E_c / E})$$

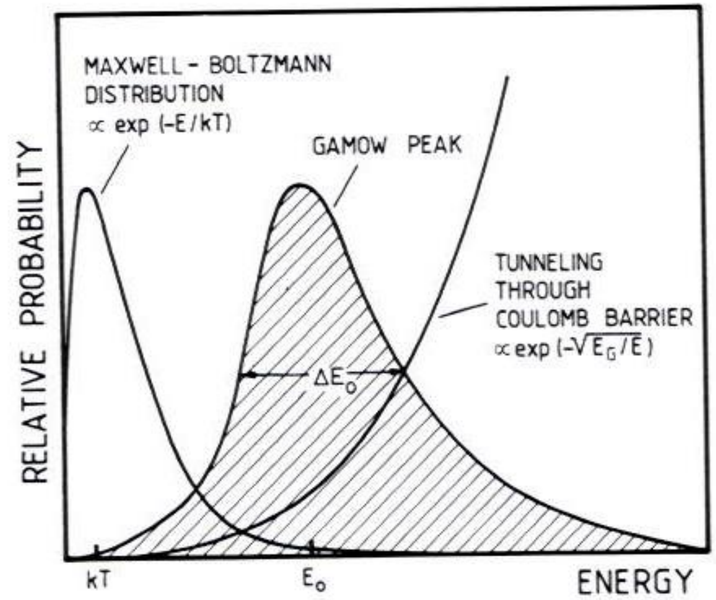
- It is convenient to consider S(E), which is slowly varying with energy , and not σ, which varies by order of magnitudes for small variations of the energy
- For each reaction, the astrophysical factor expresses the strength of the interaction
- In the table the values of S(0) for weak, e.m. and strong processes correspond to the hierarchy of the interactions
- For the p+p reaction the value is the result of theoretical calculations
- The other values are derived from extrapolations to zero energy of the experimental results
- The table also shows for each reaction the Q value

  $Q=\Sigma m_{in}-\Sigma m_{fin}$

  i.e. the energy released in the reaction.

Reaction	Process	S(0) [MeV barn]	Q [MeV]
$p+p \rightarrow d+e^{+}+\nu$	debole	$4\ 10^{-25}$	0.42 MeV
$p+d \rightarrow {}^{3}He+\gamma$	e.m.	$2.5\ 10^{-7}$	5.5 MeV
${}^{3}He+{}^{3}He \rightarrow {}^{4}He+2p$	forte	5.0	12.9 MeV

The Gamow peak



- One has to insert

$$\sigma(E) = \frac{S(E)}{E}\exp(-\sqrt{E_c / E})$$

In the expression for the reaction rate

$<\sigma v> = (8/\pi m)^{1/2} (kT)^{-3/2} \int_o dE\ \sigma(E)\ E \exp(-E/kT)$

- The result is

$$<\sigma v> = (8/\pi m)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty dES(E)\exp[-E/kT - (E_c/E)^{1/2}]$$

- Note that whereas S(E) is a slowly varying functions, the two exponentials are strongly varying, and with different trends

  -the penetration term favors high energies

  -the MB distribution decreases exponetially as a function of the energy

- **The result of the product is a typical bell shaped function**.
- The region near the maximum (Gamow peak) represents the region of energies which give the most important contribution to the process.

The position of the maximum

- Assuming that S is a slowly varying function, it can be approximated by its value at the maximum, i.e. :

$$<\sigma v>=(8/\pi m)^{1/2}\frac{S(E_0)}{(kT)^{3/2}}\int_0^\infty dE\exp[-E/kT-(E_c/E)^{1/2}]$$

- In order to find the position of the maximum, one has to study the function inside the integral. Since the exponential is a monotonous function, one has to find the stationary point of the function in the exponent

$$f = (E/kT) + (E_C/E)^{1/2}$$

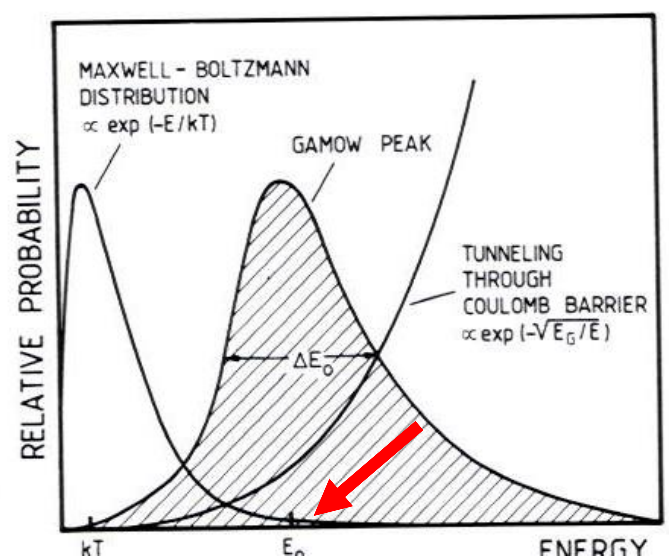
- The condition df/dE=0 determines the position of the maximum

$$E_0= (1/2\ kT)^{2/3} E_C^{1/3}$$

- Since E_C >>kT one has E_0 > kT i.e. the maximum contribution arises from energies larger than kT
- As an example, for T=1.5 10^7 ^{0}K one has kT=1.2 keV and thus :

p+p	:	E_0=5.9keV
α+^{12}C	:	E_0=56 keV
^{16}O+ ^{16}O	:	E_0=237 keV

- Note that E_0 grows rapidly with Z



The height of the peak

$$< \sigma v >= (8/\pi m)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \int_0^\infty dE \exp[-E/kT - (E_c/E)^{1/2}]$$

- The value of the integrand I(E) at the maximum, I_{max}, is obtained from the value of f for E= E_0:

  I_{max}= exp(-f(E_0)= exp(-3 E_0 /kT)
- The temperature dependence is through the variable

  $\tau = 3E_0/KT = 3/2^{2/3} (E_C/kT)^{1/3}$
- In the same conditions as before (T=1.5 10^7 ^{0}K ossia kT=1.2 keV)

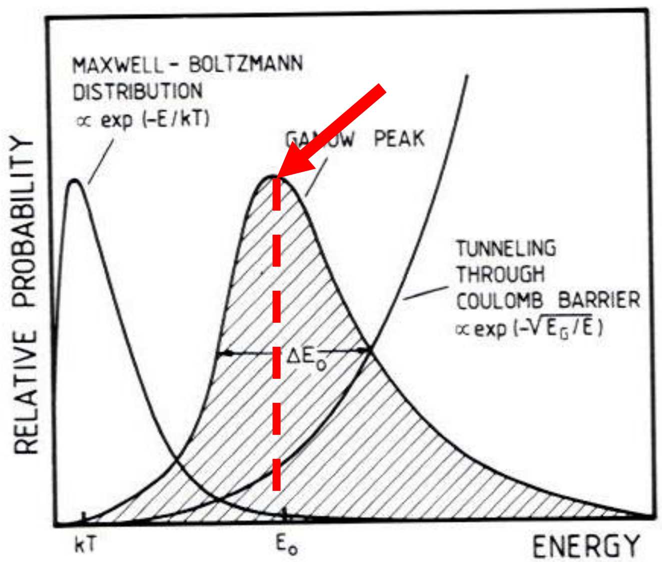
p+p : E_0=5.9keV ; I_{max}= 1.1 10^{-6}

α+^{12}C: E_0=56 keV; I_{max}= 3 10^{-57}

^{16}O+ ^{16}O: E_0=237 keV; I_{max}= 6 10^{-239}

- Since <σv> is proportional to I_{max}, these numbers show the hierarchy in the energy production rates

- First one burns lightest elements
- When these are exhausted, the star contracts, temperature increases and burning of heavier elements can start



The width of the peak

$$< \sigma v >= (8/\pi m)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \int_0^\infty dE \exp[-E/kT - (E_c/E)^{1/2}]$$

- Every bell shaped funtion can be approximated by a gaussian, close to its maximum

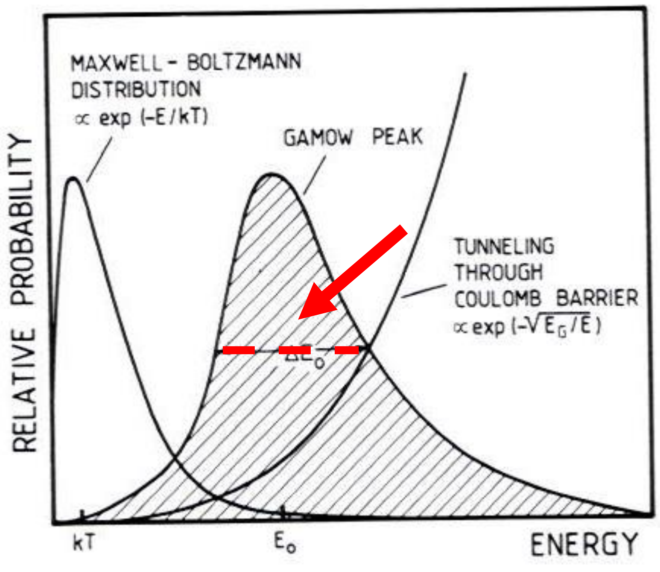
  $I(E)= I_{max} \exp[- (E-Eo)^2/\delta^2]$
- The halfwidth δ of the Gassian corresponds to the function decreasing to 1/e with respect to the maximum.
- Since I = exp[-f(E)], by expanding f arounf E_0 one has
- $f=f(E_0) +1/2 f''(E_0) (E- E_0)^2$

  This gives :

  $\delta = [-2 /f''(E_0)]^{1/2}$
- By computing the derivative one finds

  $\delta =2\ 3^{-1/2} (E_0\ kT)^{1/2}$

- I.e. δ is nearly the geometric mean between E_0 and kT
- Concerning temperature dependence, since $E_0\ \alpha\ T^{2/3}$ one has $\delta\ \alpha\ T^{5/6}$



The value of the integral

$$< \sigma v >= (8/\pi m)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \int_0^\infty dE \exp[-E/kT - (E_c/E)^{1/2}]$$

- In the gaussian approximation, $I(E) = I_{max} \exp[-(E-Eo)^2/\delta^2]$ the integral is an error function

$$\int_0^\infty dE I(E) = I_{max} \int_0^\infty dE \exp[-(E-E_0)^2/\delta^2]$$

- If $\delta << E$ on can move the lower integration limit to $-\infty$. In this way one has: $I_{max}\delta\sqrt{\pi} = \sqrt{\pi}\delta \exp(-3E_o/kT)$

- This gives the final expression for the reaction rate

$$< \sigma v > \cong 2(2/m)^{1/2} S(E_o) \frac{\delta}{(kT)^{3/2}} \exp(-3E_o/kT)$$

- This can be checked, at least for dimensions

- Let us summarize the main points :
- i) the contribution to the reaction rate comes from energies near E_o ;
- ii)this is the region to be accessed in the laboratory for the experimental study of the reaction
- iii) The reaction rate of thermonuclear reactions is strongly dependent on temperature

The reaction rate as a function of temperature, part I

$$<\sigma v> \cong 2(2/m)^{1/2} S(E_o) \frac{\delta}{(kT)^{3/2}} \exp(-3E_o/kT)$$

$$\delta = \frac{2}{\sqrt{3}}(E_o kT)^{1/2}$$

$$E_o = 2^{-2/3}(kT)^{2/3} E_c^{\;1/3}$$

- Every function y(x) for small variations of x around x_o can be expressed by a power law :

  $y=y(x_o)\,(x/x_o)^{\alpha}$.
- For determing the coefficient α let us observe that by deriving both sides :

  $dy/dx = \alpha\,(y/x)$
- This means that α is the logarithmic derivative at x_o:

  $\alpha=(dy/dx)x/y=d\,\mathit{ln}y/d\,\mathit{ln}x$
- The temperature dependence of $<\sigma v>$ is mainly through the exponential term.This means

$\ln<\sigma v> = \text{cost} - 3E_o/kT = \text{cost} - 3AT^{-1/3}$

- It follows that

$\alpha = T\,d(\ln<sv>)/dT = AT = E_o/kT$

- Therefore the dependence of rates form temperature is given by

  $\alpha = E_o/kT$
- For the usual examples (kT=1.2 keV)

p+p : E_0=5.9keV α_{pp}=4.9 ;
α+^{12}C: E_0=56 keV ; α_C=47
^{16}O+ ^{16}O: E_0=237 keV α_O=20

*

The reaction rate as a function of temperature, part II

$$<\sigma v> \cong 2(2/m)^{1/2} S(E_o) \frac{\delta}{(kT)^{3/2}} \exp(-3E_o/kT)$$

$$\delta = \frac{2}{\sqrt{3}} (E_o kT)^{1/2}$$

$$E_o = 2^{-2/3} (kT)^{2/3} E_c^{1/3}$$

- We have just seen that reaction rates, near temperature T can be parametrized as

  $$<\sigma v> = c\, T^{\alpha}$$

- Where the α coefficient is approximately given by

  $$\alpha = E_o/kT = c'\, T^{-1/3}$$

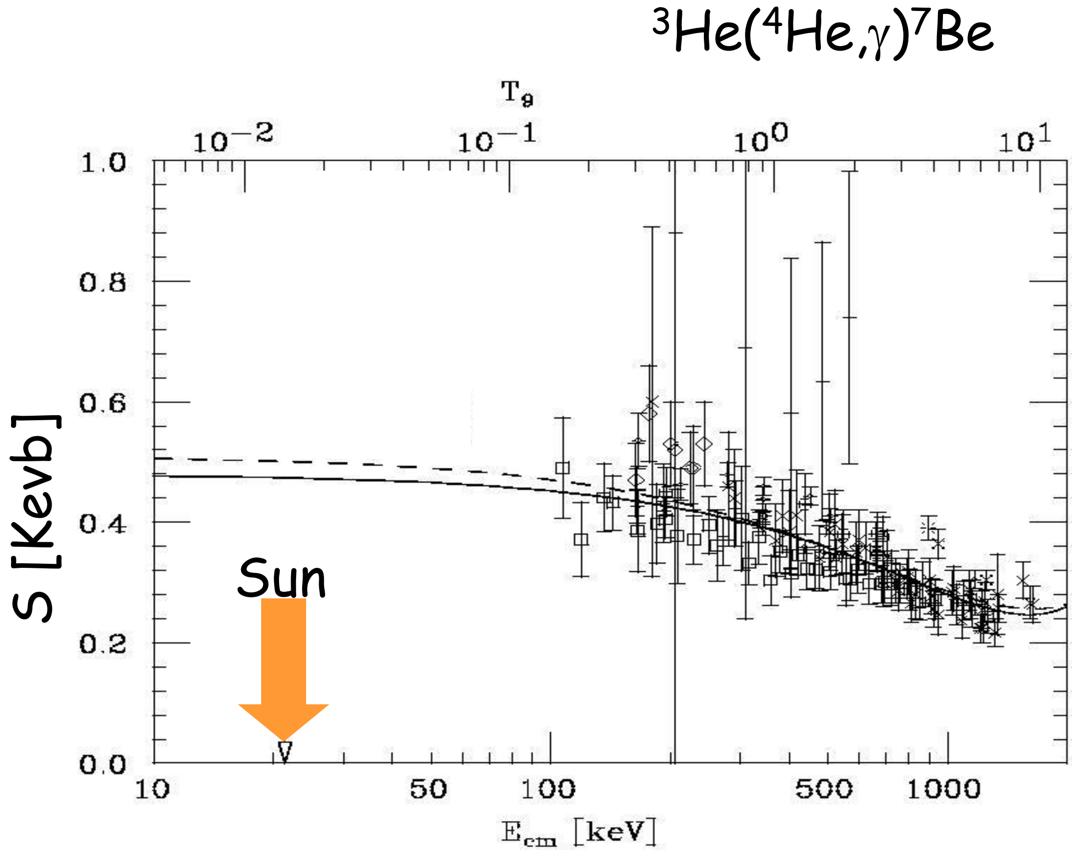
- Since α is slowly varying with temperature, α characterizes the reaction
- Note that α coefficients increase together with the charge of the reacting nuclei and can be very large
- This means that small temperature variations can produce large variations of the burning rate, which has several consequences

- i) a star starts burning a specific element only when it has reached a defined temperature (ignition)
- ii) Stars are weakly sensitive to the vales of S, since the efficiency of a reaction depends mainly on T
- iii) if one measures the reaction rate in a star (as an example, by looking at produced neutrinos) one is measuring temeprature in the production region

Experimental Determination of the astrophysical S- factor

- Nuclear physics is summarized in S(E), which (in absence of resonances) is a smooth function of E

- What matters is the value of S at or near the Gamow peak Eo
- The measurement near the Gamow peak is generally impossible and one has to extrapolate data taken at higher energies.



The lowest energies frontier

- Significant effort has been devoted for lowering the minimal detection energy
- Since counting rates become exponentially small, cosmic ray background is a significant limitation.
- This has been bypassed by installing accelerators deep underground* (The LUNA project at Gran Sasso)

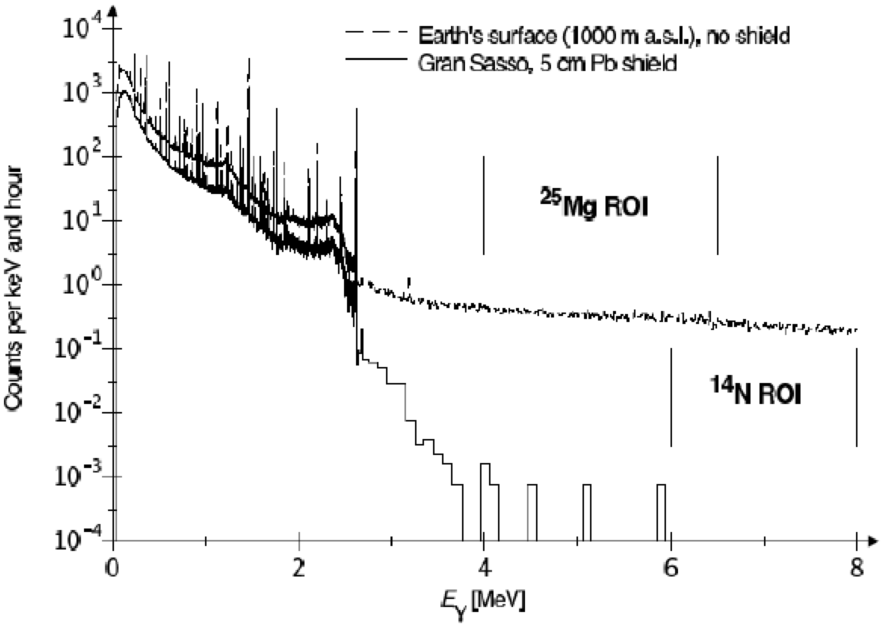
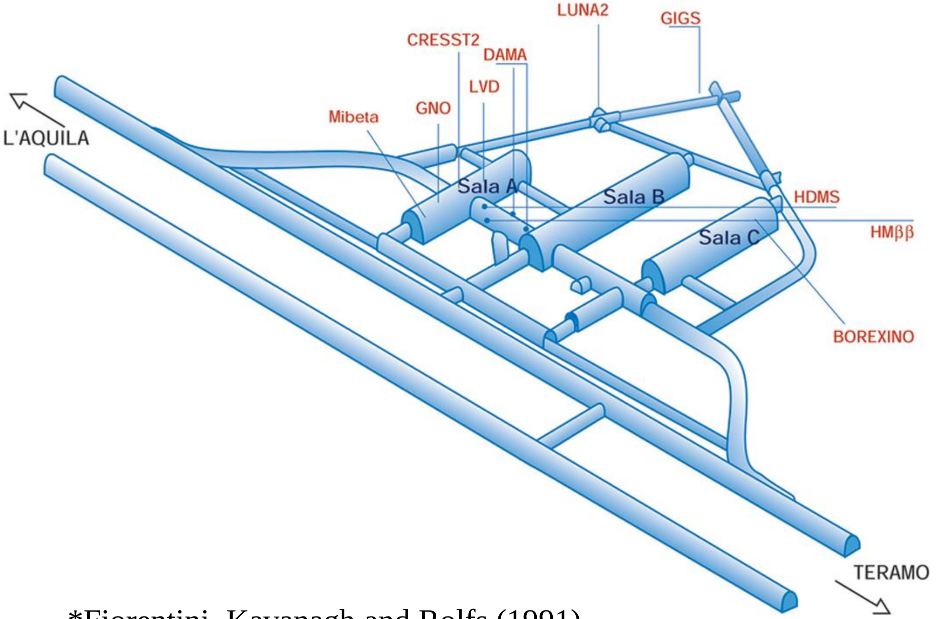


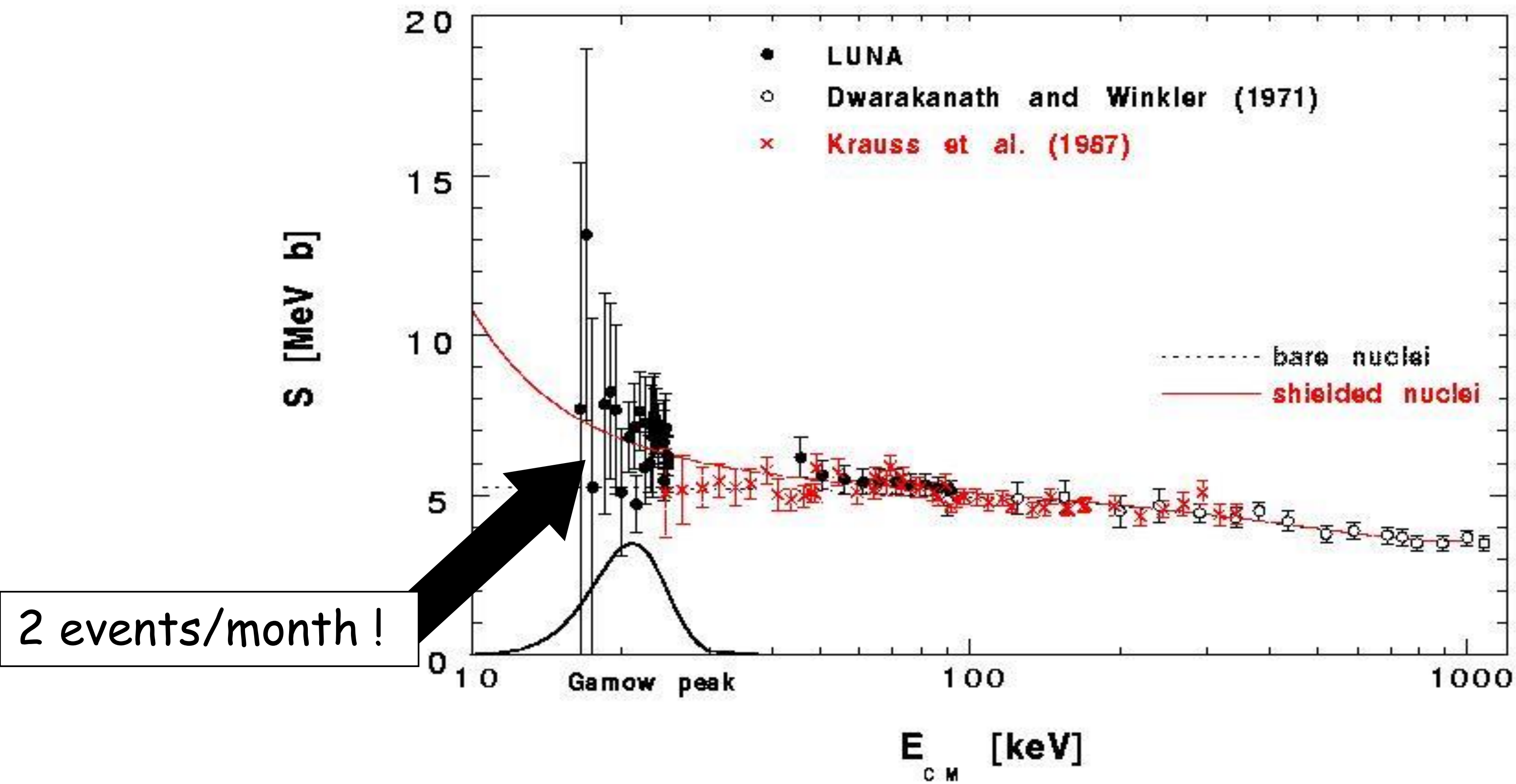
Fig. 2. Laboratory γ background as seen with the germanium detector of setup A at the earth's surface (1000 m above sea level) and inside the Gran Sasso underground facility.



*Fiorentini, Kavanagh and Rolfs (1991)

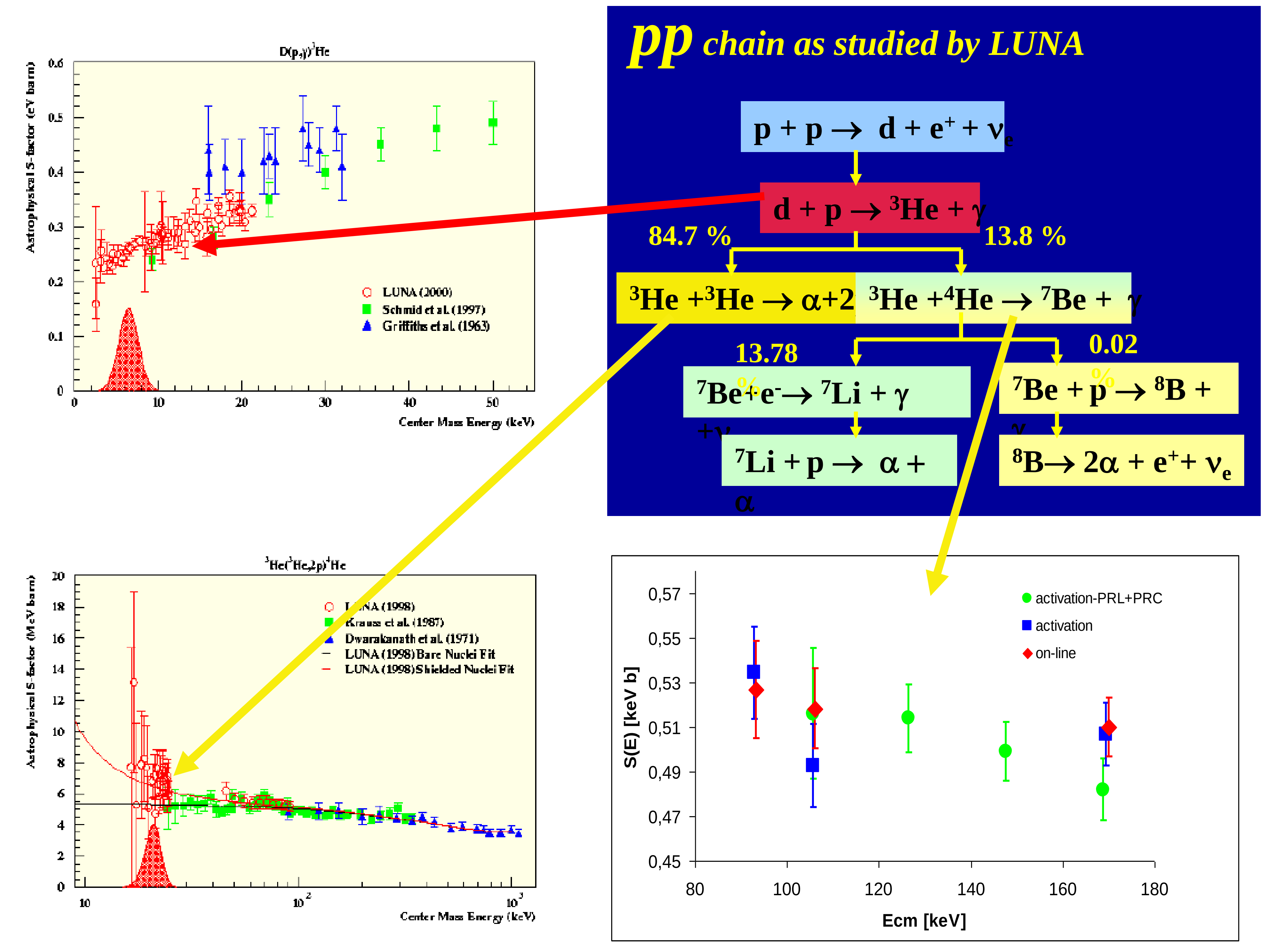
One of LUNA results*

- LUNA at LNGS has been able to measure $^3He+^3He$ at solar Gamow peak.



S(0)=5.32 (1± 6%)MeVb

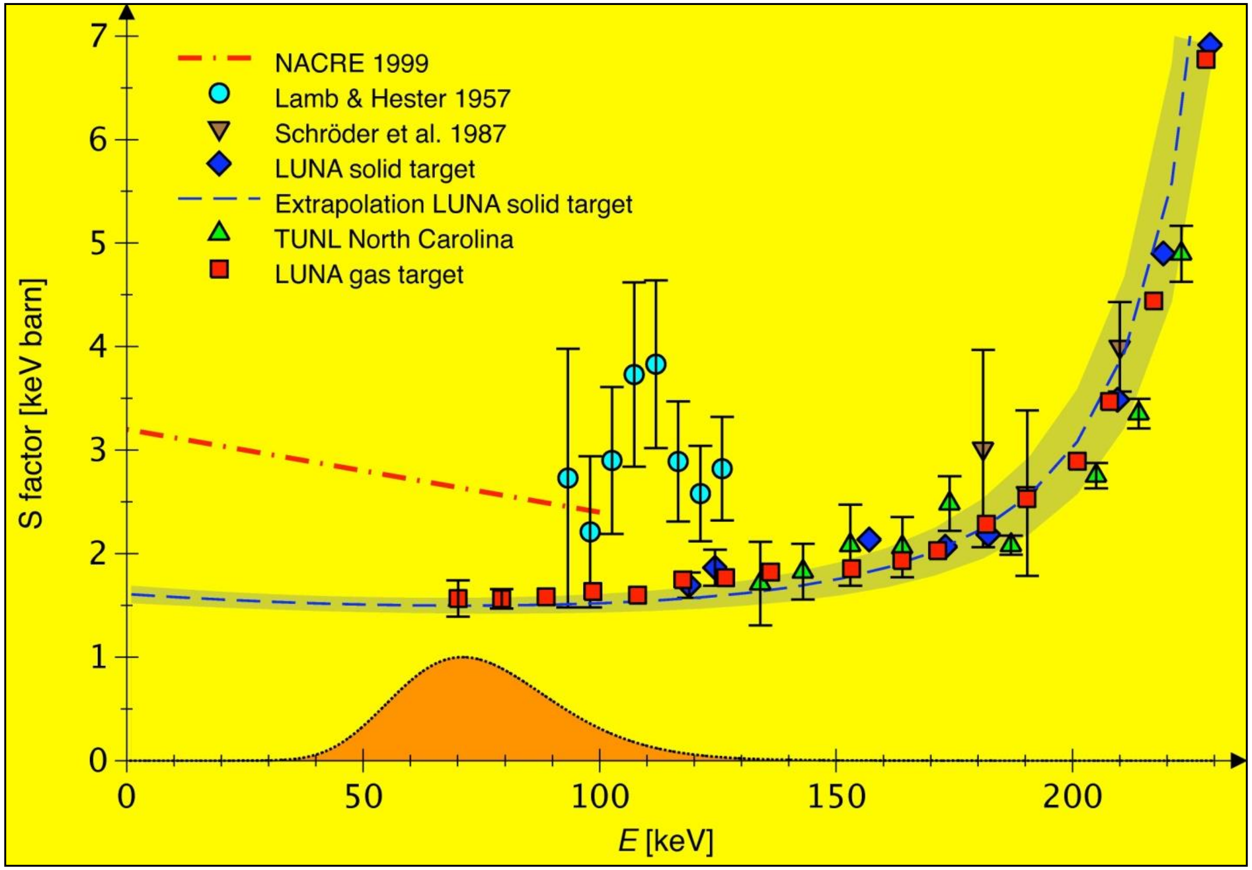
*PRL 82(1999) 5205



pp chain as studied by LUNA
p + p → d + e+ + νe
d + p → 3He + γ
84.7 %
13.8 %
3He +3He → α+2p
3He +4He → 7Be + γ
13.78 %
0.02 %
7Be+e- → 7Li + γ +ν
7Be + p → 8B + γ
7Li + p → α + α
8B→ 2α + e+ + νe
D(p,γ)3He
Astrophysical S-factor (eV barn)
Center Mass Energy (keV)
LUNA (2000)
Schmid et al. (1997)
Griffiths et al. (1963)
3He(3He,2p)4He
Astrophysical S-factor (MeV barn)
Center Mass Energy (keV)
LUNA (1998)
Krauss et al. (1987)
Dwarakanath et al. (1971)
LUNA (1998) Bare Nuclei Fit
LUNA (1998) Shielded Nuclei Fit
activation-PRL+PRC
activation
on-line
S(E) [keV b]
Ecm [keV]

CNO: LUNA results on p+ ^{14}N

- p+ ^{14}N -> 15O + γ is the key reaction governing the CNO cycle in the Sun.
- Note that the extrapolation from recent LUNA results is a factor ½ with respect to previous estimates



Appendice 1: calcolo di integrali Gaussiani

- Due integrali sono particolarmente utili

$$\int_{-\infty}^{+\infty} dx e^{-\alpha x^2} = \sqrt{\pi / \alpha} \qquad \int_{-\infty}^{+\infty} dx x^{2n} e^{-\alpha x^2} = \left| \frac{\partial^n}{\partial \alpha^n} \sqrt{\pi / \alpha} \right|$$

- Se chiamo I il primo integrale, posso scrivere

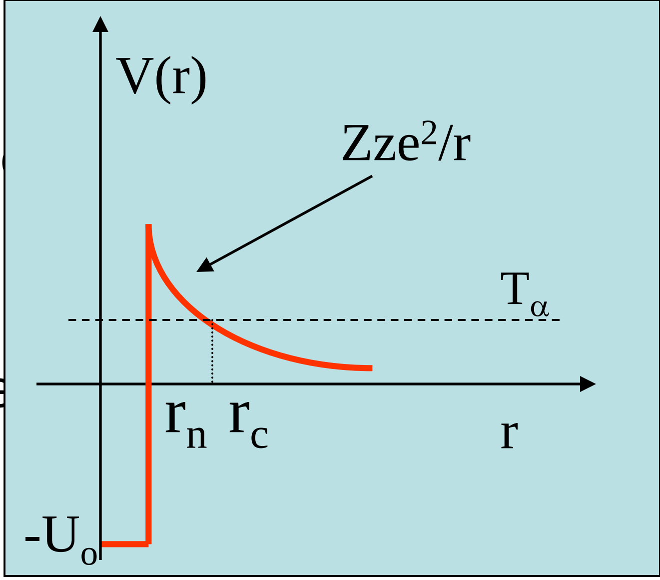
$$I^2 = \int_{-\infty}^{+\infty} dx e^{-\alpha x^2} \int_{-\infty}^{+\infty} dy e^{-\alpha y^2} = \int_{-\infty}^{+\infty} dx dy e^{-\alpha (x^2+y^2)} =$$

$$2\pi \int_{o}^{+\infty} \rho d\rho e^{-\alpha \rho^2} = \pi \int_{o}^{+\infty} dz e^{-\alpha z} = \pi / \alpha$$

Da cui I= $(\pi/\alpha)^{1/2}$

- Il secondo e' ovvio

Appendice2 : la probabilità di tunnel della particella α



$P = \exp\{-2\int_{x1}^{x2} dx[2m(V(x)-E)]^{1/2}/\hbar\}$

- Il moto radiale della particella α può essere trattato come un moto unidimensionale, da cui :

  $P = \exp\{-2\int_{rn}^{rc} dr\,[2m_\alpha\,(V(r)-T_\alpha)]^{1/2}/\hbar\}=$
  $= \exp\{-2\int_{rn}^{rc} dr\,[2m_\alpha\,(V(r)-V(r_c)]^{1/2}/\hbar\}.$

- Esplicitando $V= Zze^2/r$ e trasformando la variabile di integrazione in $x=r/r_c$ si ha:
  $P=\exp\{-2\,(2m_\alpha Zze^2 r_c/\hbar^2)^{1/2}\int_{rn/rc}^{1} dx\,[1/x-1]^{1/2}\}.$
- Osservando che che $r_n \ll r_c$ si può approssimare l'estremo inferiore con 0.

- Poiché $\int_0^1 dx\,[1/x-1]^{1/2}=\pi/2$ si trova $P= \exp\{-2\,(2m_\alpha Zze^2 r_c/\hbar^2)^{1/2}\,\pi/2\}$ da cui:

  $P=\exp\{-2\pi\, v_o/v_\alpha\}$

  dove:
  - $v_o= Zze^2/\hbar$ è la scala di velocità del problema coulombiano con cariche Z e z
  - $v_\alpha =(2T_\alpha/m_a)^{1/2}$ è la velocità finale della particella α
- Da osservare che P dipende esponenzialmente da $1/v_\alpha$ e dunque esponenzialmente da $1/\sqrt{T_\alpha}$.
- Ponendo $\lambda=\nu P$, usando la (1) e passando al logaritmo, si trova la legge di Geiger Nuttal, ($\ln\lambda=\alpha-\beta/\sqrt{T_\alpha}$) e si determinano i coefficienti ($\alpha=\ln\nu$, $\beta=2\pi zZe^2M/2^{1/2}/\hbar$) in accordo con i dati sperimentali.